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Abstract

Performing simultaneously two transformations on functions of space and time (for instance a Fourier transform on the space variable and a Laplace transform on the time variable) is easier than performing them one after the other when the variables are combined into particular invariant quantities. This is naturally also true when performing two inverse transforms simultaneously, when the conjugated variables are combined into a propagator. An immediate application is found in the computation of the solutions of partial differential equations. This article contains several general examples of such "simultaneous double transforms" for arbitrary analytic functions of space and time.

Introduction

Let us consider a function f depending on r, the distance to the origin in $\mathbf{R}^d$, and the time $t \in \mathbf{R}^+$ and suppose that Fourier and Laplace transforms of$f$ exist. We denote by k the Fourier conjugated variable, and by s the Laplace parameter. We write $f(r) \triangleright_d \widehat{f}(k)$ and $f(t) \sqsupset F(s)$ the d-dimensional Fourier transformation (we give a rigorous definition of it in Section 1) and Laplace transformation pairs, respectively. The Fourier-Laplace transform of f, that we denote by $\widehat{F}$, is often used to solved partial differential equations. Performing two transforms is a demanding computation: To solve the isotropic radiative transfer equation in two dimensions, Sato obtained for instance an intermediate step of calculation as a series of modified Bessel functions of the second kind and fractional order[8]. Such expressions require using transformation handbooks like References [1, 4, 3, 7] or formal computation softwares which obfuscates the

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computation process. We show that Sato could have used the straightforward formula

$$\frac{1}{2\pi}\frac{f\left(\sqrt{t^2-r^2}\right)}{\sqrt{t^2-r^2}}\Theta(t-r) \quad \underset{2}{\sqsupset\!\triangleright} \quad \frac{F\left(\sqrt{k^2+s^2}\right)}{\sqrt{k^2+s^2}} \tag{1}$$

and discuss the existence and applicability of similar relations in arbitrary dimensions. The advantage of an expression such as (1) is that only one transformation appears in the final result and consequently the analytic computation of the Fourier-Laplace transform or inverse transform of functions of the form (1) represent a computational simplification. We show indeed that using (1), the solution of the isotropic radiative transfer in two dimension is obtained after handling only elementary functions and Laplace transforms.

In section 1 we define the Fourier transform of a function depending on the distance r to the origin in $\mathbf{R}^d$ and show how it is related to the isotropic Green's function in d dimensions. In section 2 we demonstrate the general form of simultaneous double transformations and provide a table of such transformations. We illustrate the application of formula (1) to the isotropic radiative transfer equation in section 3 and discuss our results.

I Isotropic Fourier transforms in d dimensions

The laws of ballistics state that an object starting to move at $t = 0$ at constant speed $c = 1$ in the constant direction $\hat{\boldsymbol{u}} \in \mathcal{S}^d$ ($\mathcal{S}^d$ is the unit sphere of $\mathbf{R}^d$) will find itself at the position $\boldsymbol{r} = t\hat{\boldsymbol{u}}$ at time $t > 0$. We can therefore define the ballistic Green's function

$$g_d(\boldsymbol{r},\, t,\, \hat{\boldsymbol{u}}) = \delta^{(d)}\left(\boldsymbol{r} - t\hat{\boldsymbol{u}}\right), \tag{2}$$

that is the distribution over space and time of the object. The *isotropic* (or radial) delta-function is the average of g_d over all the spatial directions $\hat{\boldsymbol{u}}$

$$h_d(r,\, t) = \frac{1}{S_d}\int_{\mathcal{S}^d} g_d(\boldsymbol{r},\, t,\, \hat{\boldsymbol{u}})\mathrm{d}\hat{\boldsymbol{u}} = \frac{\delta(r-t)}{S_d\, r^{d-1}}, \tag{3}$$

where $S_d = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ denotes the measure of the unit sphere $\mathcal{S}^d$ in $\mathbf{R}^d$. We define the Fourier transform of a function f as

$$\widehat{f}(\boldsymbol{k}) = \int_{\mathbf{R}^d} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}. \tag{4}$$

The Fourier transform of g_d is therefore

$$\widehat{g}_d(\boldsymbol{k},\, t,\, \hat{\boldsymbol{u}}) = \exp\left(-\mathrm{i}t\boldsymbol{k}\cdot\hat{\boldsymbol{u}}\right). \tag{5}$$

One observes in the expression (5) that the directional average of $\widehat{g}_d(\boldsymbol{k},\, t,\, \hat{\boldsymbol{u}})$ only depends on the product kt where $k = \|\boldsymbol{k}\|$. The integration of this expression over $\hat{\boldsymbol{u}}$ yields

$$\widehat{h}_d(k,\, t) = \frac{(2\pi)^{d/2}}{S_d}\,(kt)^{1-\frac{d}{2}}\, J_{\frac{d}{2}-1}(kt) \tag{6}$$

where J_ν denotes the Bessel function of order ν defined in Ref. [1, Eq. 9.1.20]. In usual space dimensions, we have

$$\widehat{h}_1(k, x) = \cos(kx), \quad \widehat{h}_2(k, x) = J_0(kx), \quad \widehat{h}_3(k, x) = \frac{\sin(kx)}{kx}. \tag{7}$$

An isotropic, or spherically symmetric, function f defined on $\mathbf{R}^d$ depends on $r = \|\boldsymbol{r}\|$ only, therefore there exists a function ψ such that for all $\boldsymbol{r} \in \mathbf{R}^d$, $f(\boldsymbol{r}) = \psi(\|\boldsymbol{r}\|)$. We will abusively note $f(r)$ the expression $\psi(r)$ and observing that $\widehat{f}(\boldsymbol{k})$ also depends only on $k = \|\boldsymbol{k}\|$, we will abusively denote by $\widehat{f}(k)$ the expresion $\chi(k)$, where χ is the function such that $\widehat{f}(\boldsymbol{k}) = \chi(\|\boldsymbol{k}\|)$ for all $\boldsymbol{k} \in \mathbf{R}^d$. The Fourier transform of f is

$$\widehat{f}(\boldsymbol{k}) = \int_0^\infty f(r)\, r^{d-1}\, \mathrm{d}r \int_{S^d} \mathrm{e}^{-\mathrm{i}r\boldsymbol{k}\cdot\hat{\boldsymbol{u}}} \mathrm{d}\hat{\boldsymbol{u}} = S_d \int_0^\infty f(r)\, r^{d-1} \widehat{h}_d(\|\boldsymbol{k}\|, r)\, \mathrm{d}r. \tag{8}$$

The inverse Fourier transform is obtained by a similar integral and is given by

$$f(r) = \frac{S_d}{(2\pi)^d} \int_0^\infty \widehat{f}(k)\, k^{d-1} \widehat{h}_d(k, r)\, \mathrm{d}k. \tag{9}$$

The applications $r \mapsto f(r)$ and $k \mapsto \widehat{f}(k)$ are related by the integrals (8) and (9) which kernels depend on the dimension d. These applications form a d-dimensional isotropic Fourier pair that we denote by $f(r) \triangleright_d \widehat{f}(k)$.

II Efros's composition and double transforms

The simultaneous Fourier-Laplace double transformation is based on a Laplace pair

(i) which Laplace original is of the form of Equation (6);

(ii) that is suitable for the generalized convolution discovered by Efros[2, 5].

The condition (i) requires that the Laplace original the Fourier transform of an isotropic delta-function (3). The general form of the Laplace pair is

$$\widehat{h}_d(k,\, \tau(t,\, u)) \quad \sqsupset \quad \psi(k,\, s)\mathrm{e}^{-u\varphi(k,\, s)}, \tag{10}$$

where u is a positive real number, $\tau(t,\, u)$ is a real function such that $\tau(t,\, u) > 0$ for any t and u and $\varphi(k,\, s)$ is a function such that $\mathrm{Re}(\varphi(k,\, s)) > 0$ for large enough values of $\mathrm{Re}(s)$. There is no further condition on the function $\psi(k,\, s)$. Let me introduce here an example of a relation like Equation (10) with $d = 2$ as an illustration to this rather formal condition. This example is found in Ref. [1, Eq. 29.3.92]:

$$J_0\left(k\sqrt{t^2 - u^2}\right) \quad \sqsupset \quad \frac{1}{\sqrt{s^2 + k^2}}\mathrm{e}^{-u\sqrt{s^2+k^2}}. \tag{11}$$

Performing the Fourier inversion of the left-hand member of Equation (10), we obtain the simultaneous double transformation pair

$$\frac{\delta\left(r-\tau(t,\,u)\right)}{S_d\,r^{d-1}} \quad \underset{d}{\sqsupset\!\triangleright} \quad \psi(k,\,s)\mathrm{e}^{-u\varphi(k,\,s)}. \tag{12}$$

Carrying on the example of Equation (11), one obtains the two-dimensional simultaneous double transformation pair

$$\frac{1}{2\pi r}\delta\left(r-\sqrt{t^2-u^2}\right) \quad \underset{2}{\sqsupset\!\triangleright} \quad \frac{1}{\sqrt{s^2+k^2}}\mathrm{e}^{-u\sqrt{s^2+k^2}}. \tag{13}$$

Let us now multiply the Equation (12) by an arbitrary analytic function $f(u)$ and integrate over the variable u. The left-hand side becomes a simple integration with a Dirac delta function and the right-hand side is the Laplace transform of f for the conjugated variable $\varphi(k,\,s)$:

$$\frac{1}{S_d\,r^{d-1}}\int_0^\infty \delta(r-\tau(t,u))f(u)\,\mathrm{d}u \quad \underset{d}{\sqsupset\!\triangleright} \quad \psi(k,\,s)F\left(\varphi(k,\,s)\right). \tag{14}$$

In our example, the function $\tau(t,u)=\sqrt{t^2-u^2}$ takes the value r only if $t>r$ and for a unique $u>0$ equal to $u_1=\sqrt{t^2-r^2}$. To compute the integral of the left-hand side of Equation (14) one has to take care of the integral of the Dirac-function and replace $\delta(r-\sqrt{t^2-u^2})$ by $\delta(u-u_1)/|\partial_u\tau(t,\,u_1)|\Theta(t-r)=ru_1^{-1}\,\delta(u-u_1)\Theta(t-r)$. We finally obtain a formula for a simultaneous double transformation pair of remarkable symmetry:

$$\frac{f\left(\sqrt{t^2-r^2}\right)}{2\pi\sqrt{t^2-r^2}}\Theta(t-r) \quad \underset{2}{\sqsupset\!\triangleright} \quad \frac{F\left(\sqrt{s^2+k^2}\right)}{\sqrt{s^2+k^2}}. \tag{15}$$

The simultaneous double transformation reduces in Equation (15) to a *single* Laplace transformation.

In the general case, we have to consider, for fixed r and t, the equation $r=\tau(t,\,u)$ for the unknown u and call $u_n(r,t)$, $n=1,\,2,\,\ldots,\,N(r,t)$ the solutions of the equation. The general result follows

$$\frac{1}{S_d\,r^{d-1}}\sum_{n=1}^{N(r,t)}\frac{f\left(u_n(r,t)\right)}{|\partial_u\tau(t,u_n(r,t))|} \quad \underset{d}{\sqsupset\!\triangleright} \quad \psi(k,\,s)\;F\left(\varphi(k,\,s)\right). \tag{16}$$

Note that in the expression (15), the Θ function translates the fact that $N(r,t)=1$ if $r<t$ and $N(r,t)=0$ otherwise. Examples of Fourier-Laplace pairs obtained using Efros's composition theorem are presented in Table 1.

III Application to the isotropic radiative transfer

The radiative transfer equation in two dimensions has been solved in the isotropic case by Sato and Paasschens[8, 6]. The latter having also provided the directional elementary solution. The equation reads

$$\partial_t j(r,\,t,\,\hat{\boldsymbol{u}})-c\hat{\boldsymbol{u}}\cdot\nabla_r j(r,\,t,\,\hat{\boldsymbol{u}})+\frac{c}{\ell}j(r,\,t,\,\hat{\boldsymbol{u}})=\frac{c}{\ell}\int_{S^2}j(r,\,t,\,\hat{\boldsymbol{v}})\mathrm{d}\hat{\boldsymbol{v}} \tag{17}$$

where c is the wave celerity, ℓ the extinction length, $\hat{\boldsymbol{u}}$ is a unit vector and the function $j(\boldsymbol{r}, t, \hat{\boldsymbol{u}})$ is the radiance: The energy flowing per time unit through an area element $\mathrm{d}A$ with normal vector $\hat{\boldsymbol{u}}$ at point $\boldsymbol{r}$ and time t is $j(\boldsymbol{r}, t, \hat{\boldsymbol{u}})\mathrm{d}A$. We denote by $i(r, t)$ the directional average of $j(\boldsymbol{r}, t, \hat{\boldsymbol{u}})$. The Fourier-Laplace transform of the equation (17) is

$$\widehat{\jmath}(\boldsymbol{k}, s, \hat{\boldsymbol{u}}) - \widehat{\jmath}(\boldsymbol{k}, 0, \hat{\boldsymbol{u}}) + \mathrm{i}c\hat{\boldsymbol{u}} \cdot \boldsymbol{k}\widehat{J}(\boldsymbol{k}, s, \hat{\boldsymbol{u}}) + \frac{c}{\ell}\widehat{J}(\boldsymbol{k}, s, \hat{\boldsymbol{u}}) = \frac{c}{\ell}\int_{\mathcal{S}^2} \widehat{J}(\boldsymbol{k}, s, \hat{\boldsymbol{v}})\mathrm{d}\hat{\boldsymbol{v}} \tag{18}$$

Recognizing in Equation (2) the expression of $\widehat{G}_2(k, s, \hat{\boldsymbol{u}}) = (s + \mathrm{i}c\boldsymbol{k} \cdot \hat{\boldsymbol{u}})^{-1}$, the equation rewrites after the 2-D Fourier-Laplace simultaneous double transformation

$$\widehat{J}(\boldsymbol{k}, s, \hat{\boldsymbol{u}}) = \widehat{G}_2\left(\boldsymbol{k}, s + \frac{c}{\ell}, \hat{\boldsymbol{u}}\right)\widehat{\jmath}(\boldsymbol{k}, 0, \hat{\boldsymbol{u}}) + \frac{c}{\ell}\widehat{G}_2\left(\boldsymbol{k}, s + \frac{c}{\ell}, \hat{\boldsymbol{u}}\right)\widehat{I}(k, s), \tag{19}$$

and taking the directional average with the isotropic initial condition $j(\boldsymbol{r}, 0, \hat{\boldsymbol{u}}) = \frac{A_0}{2\pi}\delta^{(2)}(\boldsymbol{r})$, we get the expression

$$\widehat{I}(k, s) = A_0 \frac{\widehat{H}_2\left(k, s + \frac{c}{\ell}\right)}{1 - \frac{c}{\ell}\widehat{H}_2\left(k, s + \frac{c}{\ell}\right)}. \tag{20}$$

Let us use the equality $\widehat{H}_2(k, s) = (s^2 + k^2)^{-1/2}$ and formula (15) (also given as entry 2.1 of Table 1 for $d = 2$) with the function f such that $F(s) = s/(s - c/\ell)$. The inversion of F is elementary and yields $f(t) = \delta(t) + (c/\ell)\exp[ct/\ell]$. We finally obtain the result, as found in the References [8, 6],

$$i(r, t) = \frac{A_0}{2\pi}\left[\frac{\delta(r - ct)}{r} + \frac{\mathrm{e}^{\frac{1}{\ell}\sqrt{c^2t^2 - r^2}}}{\ell\sqrt{c^2t^2 - r^2}}\Theta(ct - r)\right]\mathrm{e}^{-ct/\ell}. \tag{21}$$

The leading decreasing exponential is given by an elementary property of the Laplace transform. Note that the delta function requires a particular attention to get the final result. The directional solution $j(\boldsymbol{r}, t, \hat{\boldsymbol{u}})$ is obtained using the expression (21) in the Equation (19). We have thus obtained the solution (21) by means of the general form of the 2-D Fourier-Laplace simultaneous double transformation(15) and some basic properties of the elementary function h_2.

IV Discussion

The simultaneous inversion of the space-Fourier and the time-Laplace transformations can be performed in certain particular cases, when the conjugated variables are combined in a simple way in the Fourier-Laplace domain. In these cases, only the computation of a single inversion is required to obtain the double inverse in the space-time domain. Table 1 displays a summary of the results presented in this article. Looking up at this table one can remark that in all of its entries the variables are combined in a similar way in the space-time domain and in the Fourier-Laplace domain. Such symmetry between linear combinations in space-time domain and their *conjugated combinations* in the Fourier-Laplace follows from the similarity between the Fourier transformation, the Laplace transformation and their inverse transformations. They merely reflect the fact that using an image as an original of another transformation is almost like performing an inverse transformation on this image. But we have not only obtained relations between functions of linear combinations: Consider for instance the entry 2.1 in Table 1, where the variable combination is $Q(r,t) = \sqrt{t^2 - r^2}$ in the space-time domain and $\sqrt{s^2 + k^2}$ in the conjugated space, or the entry 2.2, where these combinations are respectively $r^2/4t$ and k^2/s. These simultaneous double transformation pairs have been obtained using intermediate transforms that do not display such symmetry, it appears therefore that the simultaneous double transformations preserve, in certain cases, more symmetry than single transformations.

Some combinations of space and time that appear in the Table 1 are invariants for a physical phenomenon. The combination $t - r$ in the entries 1.1, 1.2 and 1.3 is invariant under galilean transformations, the combination r^2/t reflects the self-similarity of diffusion, and $Q = \sqrt{t^2 - r^2}$ is a proper time in special relativity. Reciprocally, the conjugated combinations are closely related to the propagators of these invariants. For instance in the application to the radiative transfer described in Section 3 the solution (20) is expressed thanks to the free propagator $\widehat{H}_2$. The entry 2.1 for $d = 2$ can be rewritten as $f(Q)/2\pi Q \sqsupset\!\!\triangleright_2 \widehat{H}_2 F(\widehat{H}_2^{-1})$. We may interprete this simultaneous double transformation as a description in terms of free propagators in the same way as the Fourier transform is a description in terms of modes.

The combinations present in Table 1 are the only ones obeying the requirements of the composition summarized in Equation (10) that can be derived from the reference tables [1, 3, 7]. They have been discovered and used because of their frequent appearance in the equations of Physics. One may nevertheless wonder whether there are not other Fourier-Laplace simultaneous double transformation pairs for these, or other, combinations (invariants or propagators). Furthermore, the question of finding new simultaneous double transformation pairs extends to any other double transformation, replacing the spatial Fourier transformation or the time Laplace transformation by other transformations taken among the large set of indexed integral transformations (Mellin transform, z-transform, Bessel transforms...).

The general principle of simultaneous double transformation can be extended to higher dimensions and other transforms. It may prove useful in other physical sciences where partial differential equations are important. Double transformations will hopefully help solving more partial differential equations in an easier way.

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Table 1: Table of simultaneous Fourier-Laplace double transformations for spherically symmetric function in space dimension d. These transforms are based on the Equation (16) applied to several identities givern in reference. The first index distinguishes between 1) simple multiplication of $F(s)$ and 2) more complicated arguments of F. The dots at the end of some lines indicate that the expression extends to the next line.

#	**Fourier-Laplace domain**	**Space-time domain**	References and notes
1.1	$(s^2+k^2)^{\frac{1-d}{2}} F(s)$	$\frac{\pi S_{d-1}}{(2\pi)^d} \frac{1}{r} f(t-r)\Theta(t-r)$	$d \geq 2$ Ref. [7], eq. 1.1.5.30
1.2	$\frac{(s+\sqrt{s^2+k^2})^{1-d/2}}{\sqrt{s^2+k^2}} F(s)$	$\frac{1}{(2\pi r)^{d/2}} f(t-r)\Theta(t-r)$	Ref. [7], eq. 1.1.5.28
1.3	$\left(s+\sqrt{s^2+k^2}\right)^{1-d/2} F(s)$	$\frac{\frac{d}{2}-1}{(2\pi)^{d/2}} \frac{1}{r^{\frac{d}{2}+1}} f(t-r)\Theta(t-r)$	$d \neq 2$ Ref. [7], eq. 1.1.5.29
1.4	$\frac{1}{s^{d/2}} \mathrm{e}^{-k^2/4s} F(s)$	$\frac{1}{\pi^{d/2}} f(t-r^2)\Theta(t-r^2)$	Ref. [7], eq. 1.1.5.32
1.5	$\exp\left[-a(\sqrt{s^2+k^2}-s)\right] \ldots$	$\frac{1}{(2\pi)^{d/2}} \frac{(a+\sqrt{r^2+a^2})^{1-d/2}}{\sqrt{r^2+a^2}} f\left(t+a-\sqrt{r^2+a^2}\right) \ldots$	
	$\times \frac{(s+\sqrt{s^2+k^2})^{1-d/2}}{\sqrt{s^2+k^2}} F(s)$	$\times \Theta\left(t+a-\sqrt{r^2+a^2}\right)$	Ref. [7], eq. 1.1.5.31
2.1	$\frac{(s+\sqrt{s^2+k^2})^{1-d/2}}{\sqrt{s^2+k^2}} F\left(\sqrt{s^2+k^2}\right)$	$\frac{1}{(2\pi)^{d/2}} \frac{(t+\sqrt{t^2-r^2})^{1-d/2}}{\sqrt{t^2-r^2}} f\left(\sqrt{t^2-r^2}\right)\Theta(t-r)$	Ref. [1], eq. 29.3.97 Ref. [7], eq. 1.1.5.34
2.2	$\frac{1}{s^{d/2}} F\left(\frac{k^2}{s}\right)$	$\frac{1}{(2\pi)^{d/2}} \frac{r^{2-d}}{(2t)^{2-d/2}} f\left(\frac{r^2}{4t}\right)$	Ref. [1], eq. 29.3.80 Ref. [7], eq. 1.1.5.27
2.3	$\frac{(s+\sqrt{s^2+k^2})^{1-d/2}}{\sqrt{s^2+k^2}} F\left(\sqrt{s^2+k^2}-s\right)$	$\frac{1}{(2\pi)^{d/2}} \frac{r^{2-d}}{t^{2-d/2}} f\left(\frac{r^2-t^2}{2t}\right)\Theta(r-t)$	Ref. [3], eq. 4.15(21)
2.4	$\frac{1}{s^{d/2}} F\left(s+\frac{k^2}{4s}\right)$	$\frac{1}{(2\pi)^{d/2}} \frac{1}{t+\sqrt{t^2-r^2}} f\left(t-\sqrt{t^2-r^2}\right)\Theta(t-r)$	Ref. [7], eq. 1.1.5.38